

Two-sided radial SLE and length-biased chordal SLE

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Abstract

We show that, for $\kappa \leq 4$, the integral of the two-sided radial SLE_κ measures over all interior points is chordal SLE_κ biased by the path's natural length, which is its $(1 + \frac{\kappa}{8})$ -dimensional Minkowski content.

1 Introduction

The present paper establishes a relationship between chordal SLE paths and a certain aggregate measure of two-sided radial SLE paths in simply connected planar domains. To explain the motivation and context for our results, we will first mention Brownian measures on curves.

The study of loop measures has flourished since it became apparent that Brownian loop measures and loop soups play a major role in the theory of conformal invariance [15, 19, 13]. Loop measures can be defined algebraically in the setting of Markov chains, and at this level the relationship between loop-erased random walk, uniform spanning trees, loop measures and Gaussian free fields is already apparent [7]. However, in the continuous plane, the Brownian loop measure gains extra significance from its conformal invariance property. Thanks to Lawler and Trujillo Ferreras' proof that the Brownian loop measure is the scaling limit of random walk loop measure [12], the discrete and continuum loop measures can profitably be used together, as seen in the identification of the scaling limit of the Green's function for loop-erased random walk [1].

In the proof that the Brownian loop measure is conformally invariant [13], it is very important that the loops are parametrized *naturally*. In the case of

Brownian motion, this means that the quadratic variation of the path up to any time t is exactly t . The general principle for the construction of unrooted loop measures, seen both discretely and in the continuum [12, 13], is that if we take the integral of rooted measures over all possible roots, we have overcounted the true unrooted measure by a factor of the random curve's length. (Heuristically, we want to count each curve only once, not once for each possible root.)

Motivated by the relationship seen among Brownian measures, we dedicate this paper to the proof of Theorem 1, which describes chordal SLE_κ as an unrooting of rooted measures in a sense that we shall now describe.

In the case of chordal SLE_κ , the measure rooted at an interior point ζ is *two-sided radial SLE_κ through ζ* , to be denoted μ_ζ , which should be thought of as chordal SLE_κ restricted to those curves passing through the interior point ζ (and normalized appropriately).

The question that arises from, among other inspirations, the theory of Brownian measures on curves, is the following: how does the aggregate of rooted measures compare to the original unrooted measure?

In the underlying discrete models, it is easy to conceive of the rooted measure as the unrooted measure, *restricted* (not conditioned) to those curves that pass through ζ (so that its total mass is the probability of passing through ζ). Then, in the aggregate (sum) of the rooted measures over all possible roots, every curve is counted precisely as many times as possible points it could be rooted at, which is the (discrete, graph) length of the curve. In other words, the aggregate of rooted measures is the unrooted measure biased by path length.

With Theorem 1, we prove that this result carries over to the continuum for SLE_κ , if $\kappa \leq 4$. The aggregate of rooted measures is defined to be

$$\nu = \int_D \mu_\zeta dA(\zeta),$$

which can be understood as a Riemann integral under a natural metric on measures on curves. Then, if μ is ordinary chordal SLE_κ and D is a bounded domain with analytic boundary, ν exists and

$$\frac{d\nu}{d\mu}(\gamma) = |\gamma|,$$

where $|\gamma|$ denotes the $(1 + \kappa/8)$ -dimensional Minkowski content of γ , which is the *natural parametrization* for SLE_κ (as first introduced in [11] and more

fully analyzed in [9]). In other words, the aggregate of rooted measures is the unrooted measure (chordal SLE_κ) biased by the SLE_κ curve's length under the natural parametrization.

2 Statement of the result

Let $0 < \kappa \leq 4$, and consider the chordal SLE_κ curves defined as follows. Let U_t be a standard Brownian motion and consider the solution g_t to the half-plane Loewner equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = 0, \quad \text{where } z \in \overline{\mathbb{H}} \text{ and } a = \frac{2}{\kappa}.$$

Then, as first proved by Rohde and Schramm, there is a random simple curve $\gamma : (0, \infty) \rightarrow \mathbb{H}$ such that $\gamma(0+) = 0$ and the domain of g_t is $\mathbb{H} \setminus \gamma_t$, where $\gamma_t = \gamma[0, t]$. This curve is called the chordal Schramm–Loewner evolution with parameter κ (SLE_κ) from 0 to ∞ in \mathbb{H} . If D is a simply connected domain and z, w accessible boundary points (interpreted as prime ends if necessary), we define chordal SLE_κ from z to w in D to be the image of chordal SLE_κ from 0 to ∞ in \mathbb{H} under a conformal transformation sending $\mathbb{H}, 0, \infty$ to D, z, w ; we shall ignore the curve's parametrization for now.

Chordal SLE_κ then has the following important properties:

1. *Conformal Invariance*: the image of chordal SLE_κ in D from z to w under a conformal map f is chordal SLE_κ in $f(D)$ from $f(z)$ to $f(w)$.
2. *Domain Markov Property*: given the initial segment γ_t of chordal SLE_κ in D from z to w , the remainder of the curve follows the law of chordal SLE_κ in $D_t = D \setminus \gamma_t$ from $\gamma(t)$ to w .
3. *Reversibility*: as first proved by Zhan, for $\kappa \leq 4$ the reversal of chordal SLE_κ from z to w in D is chordal SLE_κ from w to z in D , after reparametrization.

By a *configuration* we shall mean a simply connected domain D together with starting and ending points $z, w \in \partial D$, which will be suppressed from the notation.

We want to study “pinned” or “rooted” measures for SLE_κ . As mentioned in the introduction, in the case of chordal SLE_κ , the measure pinned at

an interior point ζ is *two-sided radial SLE $_{\kappa}$ through ζ* , which is (as far as the probability measure is concerned) just chordal SLE $_{\kappa}$ conditioned to pass through ζ . Its construction is much like the construction of the Brownian bridge as a Doob h -process. To understand this construction, we first need to consider the Green's function for chordal SLE $_{\kappa}$ at ζ , which is the normalized probability that the curve passes near ζ .

Let γ be chordal SLE $_{\kappa}$ in some configuration D , and let $\zeta \in D$. The (Euclidean) Green's function for chordal SLE $_{\kappa}$ is

$$G_D(\zeta) = \lim_{\epsilon \rightarrow 0+} \epsilon^{d-2} \mathbf{P}\{\text{dist}(\zeta, \gamma) < \epsilon\},$$

where $d = 1 + \frac{\kappa}{8}$ is the Hausdorff dimension of the curve. This should be thought of as the normalized probability that the curve passes through ζ . Lawler and Rezaei proved that the limit exists, and in the half-plane it takes the form

$$G_{\mathbb{H}}(\zeta) = c_{\kappa} (\text{Im } \zeta)^{d-2} (\sin \arg \zeta)^{4a-1}. \quad (1)$$

Moreover, the Green's function is conformally covariant,

$$G_D(\zeta) = |f'(\zeta)|^{2-d} G_{f(D)}(f(\zeta)), \quad f \text{ conformal on } D.$$

As the definition suggests, if we stop a chordal SLE $_{\kappa}$ curve γ at time $\tau_n = \inf\{t : |\gamma(t) - \zeta| \leq e^{-n}\}$, then $M_t^n := G_{D_{t \wedge \tau_n}}(\zeta)$ is a martingale. Tilting the chordal SLE $_{\kappa}$ measure μ by this martingale in the sense of Girsanov's theorem, so that

$$\frac{d\mu_{\zeta}^n}{d\mu}(\gamma_t) = G_{D_{t \wedge \tau_n}}(\zeta), \quad (2)$$

we obtain a finite measure μ_{ζ}^n of total mass $G_D(\zeta)$ on curves $(\gamma(t) : t \leq \tau_n)$ starting at p . Moreover, these tilted measures are consistent: if $m < n$, then $(\gamma(t) : t \leq \tau_m)$ has the same law under μ_{ζ}^n as under μ_{ζ}^m . Therefore, there is a limit measure μ_{ζ} called *two-sided radial SLE $_{\kappa}$ through ζ* on curves $\gamma : [0, \tau) \rightarrow \overline{D}$ that accumulate to ζ as $t \rightarrow \tau-$, where $\tau = \sup_n \tau_n < \infty$. Lawler proved endpoint continuity of μ_{ζ} in [6]: with probability one $\gamma(t) \rightarrow \zeta$ as $t \rightarrow \tau-$. Given a curve $\gamma : [0, \tau] \rightarrow \overline{D}$ sampled from μ_{ζ} (so that $\gamma(0) = z$ and $\gamma(\tau) = \zeta$), we let $\gamma[\tau, \infty)$ be a chordal SLE $_{\kappa}$ in the configuration (D_{τ}, ζ, w) , thereby providing the “other side” of the two-sided radial SLE $_{\kappa}$ curve.

We now address the parametrization for these curves, which is to be the natural parametrization for SLE $_{\kappa}$. Lawler and Rezaei proved that if we define

the natural time spent by γ in a set S to be the d -dimensional Minkowski content

$$\Theta(S) = \text{Cont}_d(\gamma \cap S) = \lim_{\epsilon \rightarrow 0+} \epsilon^{d-2} \text{Area}\{\zeta : \text{dist}(\zeta, \gamma \cap S) < \epsilon\} \quad (3)$$

then the limit exists with probability one, and

$$\mathbf{E}[\Theta(S)] = \int_S G(\zeta) dA(\zeta). \quad (4)$$

We parametrize the curves by natural parametrization, so that $\Theta(\gamma_t) = t$ for all t , and denote by $t_\gamma = \Theta(\gamma)$ the length of a curve γ . This allows us to consider the aggregate of the measures μ_ζ . To do so, we need to consider an appropriate metric on finite measure spaces. We shall use the Prokhorov metric induced by the metric on curves

$$d(\gamma, \gamma') = |t_\gamma - t_{\gamma'}| + \sup_{0 \leq s \leq 1} |\gamma(st_\gamma) - \gamma'(st_{\gamma'})|,$$

which metrizes the topology of weak convergence under d .

Then we may attempt to form the aggregate measure

$$\nu = \int_D \mu_\zeta dA(\zeta) \quad (5)$$

as a Riemann integral, so long as μ_ζ is continuous in ζ . This will be an improper Riemann integral in the sense of Definition 5, where the points z and w need to be excised because the Green's function tends to infinity there. See Section 3 for more details on this construction.

The aim of this paper is to relate the aggregate ν of two-sided radial SLE_κ to the original chordal SLE_κ measure μ . Consider the analogous discrete situation, in which μ is a measure on finite-length self-avoiding paths and the pinned measure μ_ζ is simply μ restricted to those paths that pass through ζ . Then the aggregate $\nu = \sum_\zeta \mu_\zeta$ is simply μ biased by the number of lattice sites that γ traverses. The continuous analogue of this number is the length of γ in the natural parametrization, and we prove the corresponding result:

Theorem 1. *Let $\kappa \leq 4$. Fix a configuration $D = (D, z, w)$ such that the domain D is bounded and ∂D is analytic. Let μ be chordal SLE_κ in D . Then the aggregate ν of two-sided radial SLE_κ in D as defined in (5) exists as an*

improper Riemann integral. It is equal to chordal SLE_κ biased by the curve's length in the natural parametrization:

$$\frac{d\nu}{d\mu}(\gamma) = t_\gamma.$$

Importantly, we use Zhan's result on reversibility of chordal SLE_κ in the proof. In particular, it follows from his result that two-sided radial SLE_κ and length-biased chordal SLE_κ are both reversible for $\kappa \leq 4$, and this property will be used to run the curves from the start and the end in alternation, rather than always from the start.

3 Measures on curves

In order to make sense of an integral of measures, we need to consider an appropriate metric space of measures on curves. Since the length of an SLE curve in the natural parametrization is random, we need a topology on the curves that is robust under small reparametrizations.

3.1 Parametrized curves

Let \mathcal{K} denote the set of parametrized curves (continuous paths) $\gamma : [0, t_\gamma] \rightarrow \mathbb{C}$, where $0 \leq t_\gamma < \infty$. We call t_γ the *time duration* of γ . The oscillation or modulus of continuity of γ is

$$\text{osc}_\gamma(\delta) = \sup\{|\gamma(s) - \gamma(t)| : |s - t| \leq \delta\},$$

which is continuous in δ . In particular, $\text{osc}_\gamma(\delta) \equiv 0$ if $t_\gamma = 0$.

Let $\|\cdot\|$ be the supremum norm. We can define metrics d and \tilde{d} on \mathcal{K} by

$$d(\gamma_1, \gamma_0) = |t_{\gamma_1} - t_{\gamma_0}| + \|\gamma_1 \circ s_1^* - \gamma_0 \circ s_0^*\|, \quad (6)$$

$$\tilde{d}(\gamma_1, \gamma_0) = \inf_{s_1, s_0} (\|s_1 - s_0\| + \|\gamma_1 \circ s_1 - \gamma_0 \circ s_0\|), \quad (7)$$

where the infimum is taken over all nondecreasing continuous functions s_i on $[0, 1]$ sending 0 to 0 and 1 to t_{γ_i} , the basic example of which is $s_i^*(x) = xt_{\gamma_i}$. We should think of \tilde{d} as detecting curves that are close after a small reparametrization, whereas d detects curves that are close after a small *linear* reparametrization.

The idea of allowing small reparametrizations has a long history. In particular, \tilde{d} is almost the same as the Skorokhod metric (see, e.g., [2, Chapter 3]) except that it allows any nonnegative time duration t_γ . The metric \tilde{d} coincides with the metric d_K in [5, Chapter 5], which was defined as an infimum over increasing homeomorphisms. We use the present definition as it clarifies the case $t_\gamma = 0$.

It is clear from the definition that $\tilde{d} \leq d$. There is no uniform bound in the other direction; for instance, one can consider $\gamma_n(t) = 2^n t \wedge 1$ on $[0, 1]$, for which $\tilde{d}(\gamma_m, \gamma_n) \leq 2^{-m}$ and $d(\gamma_m, \gamma_n) \geq 1/2$ if $m < n$. This example also shows that \mathcal{K} is not complete under \tilde{d} . Completeness is important in order to be able to define an integral of measures as a limit of Riemann sum approximations.

Unfortunately, \mathcal{K} is incomplete also under d , because of the presence of curves of zero length. This minor inconvenience can be avoided by considering d as a complete metric on $\tilde{\mathcal{K}} := [0, \infty) \times C[0, 1]$ where the curve $\gamma_0 \in \mathcal{K}$ is identified with the pair $(t_{\gamma_0}, \gamma_0 \circ s_0^*)$. Since all measures we consider will assign zero measure to zero-length curves, the extra pairs $(0, \tilde{\gamma}) \in \tilde{\mathcal{K}}$ with $\tilde{\gamma}$ non-constant are of no consequence.

Lemma 2. *If $\gamma_1, \gamma_0 \in \mathcal{K}$, then*

$$\tilde{d}(\gamma_1, \gamma_0) \leq d(\gamma_1, \gamma_0) \leq \tilde{d}(\gamma_1, \gamma_0) + \text{osc}_{\gamma_0}(2\tilde{d}(\gamma_1, \gamma_0)).$$

In particular, if $\gamma_0, \gamma_1, \dots \in \mathcal{K}$ then $d(\gamma_n, \gamma_0) \rightarrow 0$ if and only if $\tilde{d}(\gamma_n, \gamma_0) \rightarrow 0$.

The lemma implies that \tilde{d} is positive definite (so that it is actually a metric) and that d and \tilde{d} induce the same topology on \mathcal{K} .

Proof. Fix nondecreasing continuous functions s_i on $[0, 1]$ sending 0 to 0 and 1 to t_{γ_i} . Rescale s_1 appropriately for γ_0 by setting

$$\tilde{s}_1(t) = \frac{t_{\gamma_0}}{t_{\gamma_1}} s_1(t).$$

We have $|t_{\gamma_1} - t_{\gamma_0}| < \|s_1 - s_0\|$ and, since s_1 is continuous,

$$\begin{aligned} \|\gamma_1 \circ s_1^* - \gamma_0 \circ s_0^*\| &= \|\gamma_1 \circ s_1 - \gamma_0 \circ \tilde{s}_1\| \\ &\leq \|\gamma_1 \circ s_1 - \gamma_0 \circ s_0\| + \|\gamma_0 \circ s_0 - \gamma_0 \circ \tilde{s}_1\| \\ &\leq \|\gamma_1 \circ s_1 - \gamma_0 \circ s_0\| + \text{osc}_{\gamma_0}(\|s_0 - \tilde{s}_1\|). \end{aligned}$$

Moreover,

$$\|s_0 - \tilde{s}_1\| \leq \|s_0 - s_1\| + \|s_1 - \tilde{s}_1\| = \|s_0 - s_1\| + |t_{\gamma_1} - t_{\gamma_0}| \leq 2\|s_0 - s_1\|.$$

It follows that

$$d(\gamma_1, \gamma_0) \leq \|s_1 - s_0\| + \|\gamma_1 \circ s_1 - \gamma_0 \circ s_0\| + \text{osc}_{\gamma_0}(2\|s_0 - s_1\|).$$

The result follows by taking the infimum over all s_1, s_2 . \square

We introduced d for its completeness property, but \tilde{d} tends to be easier to estimate. One example of this relates to concatenating curves. If $\gamma, \eta \in \mathcal{K}$ with $\gamma(t_\gamma) = \eta(0)$, we define the concatenation $\gamma \oplus \eta$ by

$$t_{\gamma \oplus \eta} = t_\gamma + t_\eta, \quad (\gamma \oplus \eta)(t) = \begin{cases} \gamma(t), & 0 \leq t < t_\gamma, \\ \eta(t - t_\gamma), & t_\gamma \leq t \leq t_\gamma + t_\eta. \end{cases}$$

Then it is easy to see that

$$\tilde{d}(\gamma_1 \oplus \eta_1, \gamma_0 \oplus \eta_0) \leq \tilde{d}(\gamma_1, \eta_1) + \tilde{d}(\gamma_0, \eta_0), \quad (8)$$

but no similar estimate holds for d .

3.2 Metric spaces of finite measures

Let \mathcal{B} denote the Borel sets of \mathcal{K} under the topology induced by \tilde{d} (or equivalently d). Let $\mathcal{M}(\mathcal{K})$ denote the set of finite positive measures on $(\mathcal{K}, \mathcal{B})$. If $\mu \in \mathcal{M}(\mathcal{K})$ we write $|\mu|$ (rather than $\|\mu\|$) for the total mass of μ in this section, for the sake of legibility. If $|\mu| \neq 0$ we can write $\mu = |\mu| \mu^\#$ for a unique probability measure $\mu^\#$.

If d is either \tilde{d} or d , we define the Prokhorov metric on $\mathcal{M}(\mathcal{K})$ induced by d , which we also denote by d , as follows:

$$d(\mu, \nu) = \inf\{\epsilon : \mu(B) \leq \nu(B^\epsilon) + \epsilon \text{ and } \nu(B) \leq \mu(B^\epsilon) + \epsilon \text{ for all } B \in \mathcal{B}\},$$

where $B^\epsilon = \{\gamma \in \mathcal{K} : d(\gamma, B) < \epsilon\}$. This gives two different Prokhorov metrics d and \tilde{d} , since the neighborhood B^ϵ depends on the choice of metric. However, they induce the same topology on $\mathcal{M}(\mathcal{K})$. Indeed, the Prokhorov metric metrizes the topology of weak convergence of probability measures (see [2, Appendix III]), which is the same for d as for \tilde{d} . That they induce the same topology on finite measures then follows from the following lemma.

Lemma 3. *Let $\mu_n \in \mathcal{M}(\mathcal{K})$. Then $\mu_n \rightarrow 0$ if and only if $|\mu_n| \rightarrow 0$. If $\mu \in \mathcal{M}(\mathcal{K})$ is nonzero then $\mu_n \rightarrow \mu$ if and only if $|\mu_n| \rightarrow |\mu|$ and $\mu_n^\# \rightarrow \mu^\#$.*

Proof. It is easy to see that

$$||\mu| - |\nu|| \leq d(\mu, \nu) \leq |\mu| \vee |\nu|,$$

which implies that $\mu_n \rightarrow 0$ if and only if $|\mu_n| \rightarrow 0$. Moreover, $\mu_n \rightarrow \mu$ implies that $|\mu_n| \rightarrow |\mu|$. If $a > 1$, we have

$$d(\mu, \nu) \leq d(a\mu, a\nu) \leq a d(\mu, \nu), \quad d(\mu, a\mu) = (a - 1)|\mu|.$$

Hence, the distances $d(\mu_n^\#, \mu^\#)$ and $d(|\mu|\mu_n^\#, \mu)$ differ by a factor of at most $|\mu| \vee |\mu|^{-1}$. Moreover, $d(\mu_n, \mu)$ and $d(|\mu|\mu_n^\#, \mu)$ differ by at most

$$d(\mu_n, |\mu|\mu_n^\#) = ||\mu_n| - |\mu||.$$

Under the assumption that $|\mu_n| \rightarrow |\mu|$, therefore, $\mu_n \rightarrow \mu$ if and only if $\mu_n^\# \rightarrow \mu^\#$. \square

Another useful feature of the Prokhorov metric is additivity: we easily see from the definition that

$$d(\mu + \mu', \nu + \nu') \leq d(\mu, \nu) + d(\mu', \nu'). \quad (9)$$

3.3 Riemann integrals of measures

Since d makes $\tilde{\mathcal{K}}$ a complete separable metric space, the Prokhorov metric d also makes $\mathcal{M} := \mathcal{M}(\tilde{\mathcal{K}})$ a complete separable metric space. (We must allow the pairs $(0, \tilde{\gamma}) \in \tilde{\mathcal{K}}$ here in order for the space to be complete, but all the measures we consider will be supported on curves of positive time duration).

We shall define Riemann integrals as the limits of Riemann sums in the usual way. Definition 4 and the results that follow still apply if any complete separable metric space is used in place of $\tilde{\mathcal{K}}$.

Some care is needed for this construction because $\mathcal{M}(\tilde{\mathcal{K}})$ is not a normed space. For this reason, some of the classical approaches to vector-valued integration such as the Bochner integral (see, e.g., [20, Chapter V]) are not particularly apposite. The weaker Gelfand-Pettis integral defined using duality (developed, e.g., in [16, Chapter 3]) provides a more general theory of vector-valued integration, but we do not use this theory in any of our arguments. One reason for this is that even establishing that the measure-valued function is continuous is a major part of the work of this paper.

Definition 4. Let $R \subset \mathbb{R}^d$ be a rectangle (the product of one closed interval in each coordinate axis) and P a tagged partition of R , associating to each subrectangle $I \in P$ the sample point x_I . The mesh $\|P\|$ of a partition P is the length of the longest diagonal of any of its subrectangles. Let $f : R \rightarrow \mathcal{M}$ be a measure-valued function. The Riemann sum of f on P is

$$\mathcal{S}_f(P) = \sum_{I \in P} f(x_I) \operatorname{vol}(I).$$

The (proper) Riemann integral of f on R is the limit

$$\int_R f(x) d \operatorname{vol}(x) = \lim_{\|P\| \rightarrow 0} \mathcal{S}_f(P)$$

taken over all partitions P of R as the mesh $\|P\|$ tends to 0, if the limit exists.

In the case of a measure-valued function f on a general bounded domain D of integration, we define

$$\int_D f(x) d \operatorname{vol}(x) = \lim_{\|P\| \rightarrow 0} \mathcal{S}_{\hat{f}}(P)$$

where \hat{f} is f extended by 0 off D , provided that the limit exists.

As with the usual Riemann integral of functions, the proper Riemann integral fails to exist when the total mass of a measure-valued function has limit infinity at a point. We shall therefore need the notion of an improper integral. For our purposes, it will be simplest to present only the case where one or two points need to be removed from a (possibly unbounded) domain.

Definition 5. Let f be a measure-valued function on a (possibly unbounded) domain $D \subset \mathbb{R}^d$ and suppose $|f(x)|$ tends to infinity as x approaches the points z and w in \overline{D} . The improper integral of f on D is defined to be

$$\int_D f(x) d \operatorname{vol}(x) := \lim_{r,s,t \rightarrow 0} \int_{D \cap B(0, \frac{1}{r}) \setminus \overline{B(z,s)} \setminus \overline{B(w,t)}} f(x) d \operatorname{vol}(x),$$

if the limit exists.

In the remainder of this section we present some basic properties of Riemann integrals of continuous measure-valued functions. This material is unnecessary for the proof of Theorem 1, because in that theorem we prove directly the existence of the limit that defines the integral.

Lemma 6. *If $a_i \geq 0$ and $\mu_i, \nu_i \in \mathcal{M}$ for $i = 1, \dots, n$ with $\sum_i a_i = 1$ then*

$$d\left(\sum_i a_i \mu_i, \sum_i a_i \nu_i\right) \leq \sup_i d(\mu_i, \nu_i).$$

Proof. Let $V \in \mathcal{B}$ and $\epsilon > \sup_i d(\mu_i, \nu_i)$. As $\mu_i(V) \leq \nu_i(V^\epsilon) + \epsilon$ and $\nu_i(V) \leq \mu_i(V^\epsilon) + \epsilon$ for all i , we have

$$\sum_i a_i \mu_i(V) \leq \sum_i a_i \nu_i(V^\epsilon) + \epsilon \quad \text{and} \quad \sum_i a_i \nu_i(V) \leq \sum_i a_i \mu_i(V^\epsilon) + \epsilon. \quad \square$$

Suppose that f is a measure-valued function on a rectangle $R \subseteq \mathbb{R}^d$ that is continuous, hence uniformly continuous, with respect to d . For now we shall assume that R is a unit hypercube; this can be undone by scaling later.

Lemma 7. *Suppose $R \subset \mathbb{R}^d$ is a unit hypercube and $P \subset Q$ are nested tagged partitions of R ; that is, every subrectangle of Q lies in a subrectangle of P , but no statement is made about the sample points. Then*

$$d(\mathcal{S}_f(Q), \mathcal{S}_f(P)) \leq \text{osc}_f(\|P\|).$$

Proof. We have

$$\mathcal{S}_f(P) = \sum_{I \in P} \sum_{\substack{J \in Q \\ J \subseteq I}} f(x_I) \text{vol}(J), \quad \mathcal{S}_f(Q) = \sum_{I \in P} \sum_{\substack{J \in Q \\ J \subseteq I}} f(x_J) \text{vol}(J).$$

Write $\epsilon = \text{osc}_f(\|P\|)$. If $I \in P$ and $J \in Q$ with $J \subseteq I$, we know that $d(f(x_I), f(x_J)) \leq \epsilon$. Applying Lemma 6 with weights $\text{vol}(J)$ for $J \in Q$, we see that $d(\mathcal{S}_f(P), \mathcal{S}_f(Q)) \leq \epsilon$. \square

By scaling and taking a common refinement if necessary, we obtain:

Corollary 8. *Suppose $R \subset \mathbb{R}^d$ is a rectangle and P, Q are arbitrary tagged partitions of R . Then*

$$d(\mathcal{S}_f(Q), \mathcal{S}_f(P)) \leq 2(\text{vol}(R) \vee 1) \text{osc}_f(\|P\|).$$

Corollary 9. *If P_n are partitions of a rectangle R with $\|P_n\| \rightarrow 0$ then $(\mathcal{S}_f(P_n))$ is a Cauchy sequence in \mathcal{M} and has a limit independent of the sequence (P_n) . This limit is therefore the integral $\int_R f(x) d\text{vol}(x)$ of f on R . Moreover,*

$$d\left(\mathcal{S}_f(P), \int_R f(x) d\text{vol}(x)\right) \leq (\text{vol}(R) \vee 1) \text{osc}_f(\|P\|).$$

It is also easy to see that $|\int_R f(x) d\text{vol}(x)| = \int_R |f(x)| d\text{vol}(x)$.

4 Generalized escape estimates

In this section we shall demonstrate that chordal SLE biased by the time spent in a small target region is unlikely to retreat far from that region after first coming close to it. We derive this result from the uniform estimate for two-sided radial SLE, Theorem 1.3 in [4].

Since we aim to prove that the aggregate of two-sided radial SLE measures is length-biased chordal SLE, it is useful to introduce the following notation for length-biased chordal SLE.

Let μ be the law of a chordal SLE_κ curve γ in a fixed configuration (D, z, w) , which we suppress from the notation. As in Section 3, we consider this as a Borel measure on the space \mathcal{K} that is carried on naturally parametrized curves. If $Q \subseteq D$ is a square, or a finite union of squares, we shall denote by $\Theta(Q)$ the time γ spends in Q under the natural parametrization, as defined in (3), and let

$$d\mu_Q = \Theta(Q) d\mu,$$

which is chordal SLE_κ weighted by the natural time spent in Q . Since $\Theta(\ell) = 0$ with probability one for any line ℓ , as follows from (4), we have

$$\mu_Q = \sum_i \mu_{Q_i} \tag{10}$$

if $Q = \bigcup_i Q_i$ and no two squares Q_i, Q_j share an interior point.

As one might expect, the law $\mu_Q^\#$ of chordal SLE_κ biased by the time spent in a small square Q is close (in the sense of weak convergence) to the law $\mu_z^\#$ of two-sided radial SLE_κ through any point $z \in Q$. This estimate will be made precise in Section 6. However, the two measures are mutually singular, and if we want to make any estimates involving Radon–Nikodym derivatives, we need to restrict our attention to the sections of the curve before and after all approaches to Q .

The cut-off point that we choose for a square Q is the circle $C(Q)$ with the same center as Q and twice its diameter. (Any fixed ratio of diameters larger than 1 would have worked for this argument.) Then we consider the curve γ up to the first time ρ_Q that it hits $C(Q)$, that is,

$$\rho_Q = \tau_{C(Q)}, \quad \text{where} \quad \tau_X = \inf\{t : \gamma(t) \in X\}.$$

We shall frequently need to consider the reversed curve $\tilde{\gamma}$ defined by $\tilde{\gamma}(t) = \gamma(t_\gamma - t)$ and $t_{\tilde{\gamma}} = t_\gamma$. It is of course true that the map $\gamma \mapsto \tilde{\gamma}$ is

continuous. Reversibility of chordal and two-sided radial SLE_κ states that if γ is an SLE_κ curve in D from z to w (through ζ , if two-sided) then $\tilde{\gamma}$ is an SLE_κ curve in D from w to z (through ζ , if two-sided).

As we will also need to stop the reversed curve $\tilde{\gamma}$ before it approaches Q , we designate the corresponding stopping time by

$$\tilde{\rho}_Q = \tilde{\tau}_{C(Q)}, \quad \text{where} \quad \tilde{\tau}_X = \inf\{t : \tilde{\gamma}(t) \in X\}.$$

Note that this is a stopping time for the reversed curve $\tilde{\gamma}$ but not for γ itself. Nevertheless, the combination of the domain Markov property, reversibility and the uniformity of our escape estimates allows us to deduce the “whole-curve” escape estimates that we need.

The first estimate we give is a mild generalization of the two-sided radial escape estimate that appears in [4, Theorem 1.3].

Proposition 10. *If $\kappa \leq 4$, there exists $c < \infty$ such that the following holds. Let γ be a two-sided radial SLE_κ curve from z to w through ζ in a domain D that contains \mathbb{D} , where $z, w \in \partial D \cap \partial \mathbb{D}$ and $|\zeta| \leq 1/2$. Then for all $r \geq 1$,*

$$\mathbf{P}\{\gamma \cap r \partial \mathbb{D} \neq \emptyset\} \leq c r^{-(4a-1)/2}.$$

Proof. First consider the initial segment $\gamma[0, \tau_\zeta]$ of the two-sided radial SLE_κ curve stopped when it hits ζ . Let S be the set of crosscuts of D that are subsets of the circle $r \partial \mathbb{D}$ and let $\tilde{\gamma}$ be the line segment from z to ζ . If $\eta \in S$, Corollary 5.6 in [4] shows that

$$\mathbf{P}\{\gamma[0, \tau_\zeta] \cap \eta \neq \emptyset\} \leq c \sum_{\eta \in S} \mathcal{E}_D(\tilde{\gamma}, \eta)^{4a-1}.$$

It follows that

$$\begin{aligned} \mathbf{P}\{\gamma[0, \tau_\zeta] \cap r \partial \mathbb{D} \neq \emptyset\} &\leq c \sum_{\eta \in S} \mathcal{E}_D(\tilde{\gamma}, \eta)^{4a-1} \leq c \left(\sum_{\eta \in S} \mathcal{E}_D(\tilde{\gamma}, \eta) \right)^{4a-1} \\ &\leq c (2 \mathcal{E}_D(\tilde{\gamma}, r \partial \mathbb{D}))^{4a-1} \leq c \mathcal{E}_D(\partial \mathbb{D}, r \partial \mathbb{D})^{4a-1} \leq c e^{-\frac{1}{2}(4a-1)}, \end{aligned}$$

where the last inequality follows from the Beurling estimate.

Now we know that the reversal of two-sided radial SLE_κ in D from z to w through ζ is two-sided radial SLE_κ in D from w to z through ζ , so the same argument proves that

$$\mathbf{P}\{\tilde{\gamma}[0, \tilde{\tau}_\zeta] \cap r \partial \mathbb{D} \neq \emptyset\} \leq c r^{-\frac{1}{2}(4a-1)}.$$

Since with probability one ζ is visited exactly once by γ , so that $\tilde{\tau}_\zeta = t_\gamma \setminus \tau_\zeta$, we may add the two estimates obtained above to yield

$$\mathbf{P}\{\gamma \cap r \partial \mathbb{D} \neq \emptyset\} \leq c r^{-\frac{1}{2}(4a-1)}. \quad \square$$

The escape estimate for length-biased chordal SLE_κ is more complicated because it is not as clear exactly how the bias affects the local behaviour. We start with an estimate on the Green's function for chordal SLE_κ , which will allow us to prove that away from the square of interest, length-biased chordal SLE_κ behaves just like two-sided radial SLE_κ .

Proposition 11 (Main Estimate). *If $\kappa \leq 4$, there exists $c < \infty$ such that the following holds. Let (D, z, w) be a chordal configuration and $\zeta, \zeta' \in D$ with*

$$r := \frac{|\zeta - \zeta'|}{\text{dist}(\zeta, \partial D)} \leq \frac{1}{2}.$$

Then

$$\left| \log \frac{G_{D,z,w}(\zeta)}{G_{D,z,w}(\zeta')} \right| \leq c r. \quad (11)$$

Proof. The explicit formula 1 for the Green's function in the upper half-plane implies that

$$G_{D,z,w}(\zeta') = c_\kappa S_{D,z,w}(\zeta')^{4a-1} \text{crad}_D(\zeta')^{d-2},$$

where $\text{crad}_D(\zeta')$ is the conformal radius of D around ζ' and $S_{D,z,w}(\zeta') = \sin \arg f(\zeta')$ under any conformal map f sending D, z, w to $\mathbb{H}, 0, \infty$ respectively. Let $d = \text{dist}(\partial D, \zeta)$. Derivative and Harnack estimates for the positive harmonic function $S_{D,z,w}$ yield

$$|\nabla S_{D,z,w}(\zeta')| \leq c \frac{S_{D,z,w}(\zeta')}{d}$$

when $|\zeta' - \zeta| < \frac{1}{2}d$ and therefore

$$\left| \log \frac{S_{D,z,w}(\zeta')}{S_{D,z,w}(\zeta)} \right| \leq c r.$$

To estimate the conformal radius factor, we use the well-known characterization

$$\log \text{crad}_D(\zeta') = \mathbf{E}^{\zeta'} \log |B_\tau - \zeta'|$$

for a complex Brownian motion B started at ζ' and stopped at time $\tau = \inf\{t : B_t \notin D\}$ (see, e.g., [5, Corollary 3.34]).

Because $|B_\tau - \zeta| \geq d$ we have

$$\left| \log \frac{|B_\tau - \zeta'|}{|B_\tau - \zeta|} \right| \leq cr.$$

Furthermore, the distributions of B_τ when the Brownian motion is started from ζ and from ζ' are mutually absolutely continuous with Radon–Nikodym derivative $e^{O(r)}$, as can be seen by stopping B first on the circle centered at ζ of radius d . By applying a similarity transformation, we may assume that $d = 1$ and $\zeta = 0$. We then have $\log |B_\tau| \geq 0$ and therefore

$$e^{-cr} \mathbf{E}^0 \log |B_\tau| \leq \mathbf{E}^{\zeta'} \log |B_\tau| \leq e^{cr} \mathbf{E}^0 \log |B_\tau|.$$

Since Koebe’s 1/4-theorem implies that $\mathbf{E}^0 \log |B_\tau| \leq c$ we have that

$$|\mathbf{E}^{\zeta'} \log |B_\tau| - \mathbf{E}^0 \log |B_\tau|| \leq cr.$$

We may conclude that

$$\left| \log \frac{\text{crad}_D(\zeta')}{\text{crad}_D(\zeta)} \right| \leq cr.$$

The proposition follows from the estimates we have given on the conformal radius and $S_{D,z,w}$ factors. \square

An alternative proof of Proposition 11 proceeds by estimating the Green’s function in \mathbb{H} after applying conformal covariance and distortion estimates for the map to \mathbb{H} .

For the remaining propositions we fix a chordal configuration (D, z, w) and let μ denote chordal SLE_κ in that configuration. If $Q \subset D$ is a square and $\zeta \in Q$, let μ_ζ and μ_Q denote two-sided radial SLE_κ through ζ and chordal SLE_κ weighted by the time spent in Q respectively.

One of the most important consequences for us of this main estimate is the fact that length-biased chordal SLE_κ is essentially the same as two-sided radial SLE_κ , at least before the curve comes too close to the set whose natural parametrization is in question. To summarize the result, it states that chordal SLE_κ biased by the time spent in a square Q and stopped (on either side) before coming closer to Q than R times its diameter is mutually absolutely continuous with two-sided radial SLE through any point in Q , with Radon–Nikodym derivative uniformly of order $1 + O(1/R)$. The following proposition states the result precisely.

Proposition 12. *If $\kappa \leq 4$, there exists $c < \infty$ such that the following holds. Let Q be a square of diameter r_0 and let C be a circle with the same center as Q and radius $r \geq r_0$. Let $\zeta \in Q$. Let (D, z, w) be a chordal SLE configuration such that $C \subset \overline{D}$. In this configuration, let γ be an SLE_κ curve, considered under the probability measures $\mu_\zeta^\#$ of two-sided radial SLE_κ through ζ and $\mu_Q^\#$ of Q -length-biased chordal SLE_κ .*

Then, considered as measures on the pair $(\gamma[0, \tau_C], \tilde{\gamma}[0, \tilde{\tau}_C])$, the Radon–Nikodym derivative satisfies

$$\left| \log \frac{d\mu_Q^\#}{d\mu_\zeta^\#} \right| \leq c \frac{r_0}{r}. \quad (12)$$

Moreover, a similar estimate applies to the total masses

$$\|\mu_Q\| = \int_Q G_{D,z,w}(\zeta') dA(\zeta')$$

and $\|\mu_\zeta\| = G_{D,z,w}(\zeta)$:

$$\left| \log \frac{\|\mu_Q\|}{\|\mu_\zeta\| A(Q)} \right| \leq c \frac{r_0}{r}. \quad (13)$$

Proof. The total mass estimate (13) follows immediately from (11).

For the Radon–Nikodym derivative estimate (12), it suffices to prove the estimate considering the measures only on the initial segment $\gamma[0, \tau_C]$ and ignoring the final segment $\tilde{\gamma}[0, \tilde{\tau}_C]$. The reason for this is as follows. For chordal SLE_κ conditioned on the initial segment $\gamma[0, \tau_C]$ (and assuming that $\tau_C < \infty$, which holds with probability one under $\mu_Q^\#$ and $\mu_\zeta^\#$), the remainder of the curve $\gamma[\tau_C, t_\gamma]$ follows the law of chordal SLE_κ in the configuration $(D \setminus \gamma[0, \tau_C], \gamma(\tau_C), w)$. The reversal of $\gamma[\tau_C, t_\gamma]$ conditionally follows the law of chordal SLE_κ in the configuration $(D \setminus \gamma[0, \tau_C], w, \gamma(\tau_C))$, by reversibility of chordal SLE_κ . Weighting by $\Theta(Q)$ shows that, under $\mu_Q^\#$, the reversal of $\gamma[\tau_C, t_\gamma]$ conditionally follows the law of Q -length-biased chordal SLE_κ in the $(D \setminus \gamma[0, \tau_C], w, \gamma(\tau_C))$. A similar argument using reversibility of two-sided radial SLE_κ shows that, under $\mu_\zeta^\#$, the reversal of $\gamma[\tau_C, t_\gamma]$ conditionally follows the law of two-sided radial SLE_κ through ζ in the configuration $(D \setminus \gamma[0, \tau_C], w, \gamma(\tau_C))$. We may therefore apply the initial-segment estimate to the measures on the terminal segment $\tilde{\gamma}[0, \tilde{\tau}_C]$ *conditional* on the initial segment $\gamma[0, \tau_C]$. Multiplying the Radon–Nikodym derivatives obtained for

the measures on the initial segment $\gamma[0, \tau_C]$ and, conditional on that initial segment, on the terminal segment $\tilde{\gamma}[0, \tilde{\tau}_C]$, we may conclude that the Radon-Nikodym derivative estimate (12) applies to the measures on the pair $(\gamma[0, \tau_C], \tilde{\gamma}[0, \tilde{\tau}_C])$.

It remains to prove the estimate (12) considering the measures only on the initial segment $\gamma[0, \tau_C]$. If we consider only this segment, note that μ_Q is the same measure as chordal SLE_κ weighted by the conditional expectation

$$X := \mathbf{E}(\Theta(Q) 1\{\tau_C < \infty\} \mid \gamma[0, \tau_C]) = \int_Q G_{D \setminus \gamma[0, \tau_C], \gamma(\tau_C), w}(\zeta') dA(\zeta').$$

The integral expression given here follows from (3) and the domain Markov property for chordal SLE_κ . Moreover, two-sided radial SLE_κ μ_ζ is the same measure as chordal SLE_κ weighted by

$$Y := G_{D \setminus \gamma[0, \tau_C], \gamma(\tau_C), w}(\zeta),$$

as follows directly from the definition (2). By the main estimate (11),

$$\left| \log \frac{X}{Y A(Q)} \right| \leq c \frac{r_0}{r}.$$

Dividing by the total masses to obtain the probability measures $\mu_Q^\#$ and $\mu_\zeta^\#$, and using (13), we obtain (12) for the initial segment $\gamma[0, \tau_C]$. \square

An easy consequence is the escape estimate for Q -length-biased chordal SLE_κ *away from* Q .

Corollary 13. *If $\kappa \leq 4$, there exists $c < \infty$ such that the following holds. Let γ be a chordal SLE_κ curve from z to w in a domain D that contains \mathbb{D} , where $z, w \in \partial D \cap \partial \mathbb{D}$, weighted by the natural time spent in a square $Q \subset \frac{1}{2}\mathbb{D}$. Then for all $r \geq 1$,*

$$\mathbf{P}\{(\gamma[0, \rho_Q] \cup \tilde{\gamma}[0, \tilde{\rho}_Q]) \cap r \partial \mathbb{D} \neq \emptyset\} \leq c r^{-(4a-1)/2}. \quad (14)$$

Proof. Let $\zeta \in Q$, and apply (12) to $\mu_Q^\#$ and $\mu_\zeta^\#$ with the circle $C = C(Q)$ (so that $r = r_0$). We see that the Radon-Nikodym derivative between the two probability measures is uniformly bounded away from 0 and ∞ . Since the escape estimate was established for $\mu_\zeta^\#$ in Proposition 10, it carries over on the initial and final segments to yield (14). \square

More work is needed to establish the escape estimate for the *entire* length-biased chordal SLE_κ curve, which we state in the following theorem.

Theorem 14. *If $\kappa \leq 4$, there exists $c < \infty$ such that the following holds. Let γ be a chordal SLE_κ curve from z to w in a domain D that contains \mathbb{D} , where $z, w \in \partial D \cap \partial \mathbb{D}$, weighted by the natural time spent in a square $Q \subset \frac{1}{2}\mathbb{D}$. Then for all $r \geq 1$,*

$$\mathbf{P}\{\gamma \cap r\partial\mathbb{D} \neq \emptyset\} \leq cr^{-(4a-1)/2}.$$

The idea of the proof of Theorem 14 is to subdivide Q into 2^{2k} equal subsquares, to apply Corollary 13 to each of those subsquares, and to average the resulting estimates. It seems, however, that after dividing Q into smaller squares, we still have the problem of understanding the intermediate segments of γ that actually pass through those squares.

The observation that solves this problem is that after subdividing Q a finite number of times *that depends on γ* , there can be no escape during those intermediate segments. To make the argument precise, we need the following lemma, which states that it suffices to subdivide Q a finite (but γ -dependent, i.e., random) number of times.

Lemma 15. *Let Q be a square and γ a simple curve starting and ending outside $C(Q)$. Suppose that γ intersects a closed set η that lies outside $C(Q)$. There is a positive integer k such that, if \mathcal{Q}_k is the set of 2^{2k} equal subsquares of Q obtained by subdividing Q dyadically k times, then for each $Q' \in \mathcal{Q}_k$, either the initial segment $\gamma[0, \rho_{Q'}]$ or the terminal segment $\tilde{\gamma}[0, \tilde{\rho}_{Q'}]$ intersects η .*

Proof. Let D be the disk bounded by $C(Q)$. Since the set $S := \{t : \gamma(t) \in \overline{D}\}$ is compact, we may find finitely many disjoint time intervals I_1, \dots, I_n that cover S and such that the images $\gamma(I_1), \dots, \gamma(I_n)$ do not intersect η . Since γ is simple, there is a minimum distance $\delta > 0$ between any pair $\gamma(I_j), \gamma(I_k)$ of these images. Now consider a dyadic subdivision \mathcal{Q}_k of Q into squares of side length less than $\delta/(2\sqrt{2})$. Fix any square $Q' \in \mathcal{Q}_k$ and observe that $\text{diam } C(Q') < \delta$. It follows that the intermediate segment $\gamma[\rho_{Q'}, t_\gamma - \tilde{\rho}_{Q'}]$ cannot intersect η . For if the intermediate segment intersected η , there would be distinct time intervals I_j and I_k such that $\gamma(I_j)$ and $\gamma(I_k)$ both contained points inside $C(Q')$, implying that $\text{dist}(\gamma(I_j), \gamma(I_k)) < \delta$ which is impossible by our construction. Since γ intersects η , the initial segment $\gamma[0, \rho_{Q'}]$ or final segment $\tilde{\gamma}[0, \tilde{\rho}_{Q'}]$ (or both) must therefore intersect η . \square

Proof of Theorem 14. Let E denote the event that $\gamma \cap r \partial \mathbb{D} \neq \emptyset$. Introduce the dyadic subdivisions \mathcal{Q}_k of Q , as in the proof of Lemma 15. Let E_k denote the event that for each $Q' \in \mathcal{Q}_k$, either the initial segment $\gamma[0, \rho_{Q'}]$ or the terminal segment $\tilde{\gamma}[0, \tilde{\rho}_{Q'}]$ (or both) intersects η . This is an increasing sequence of events and by Lemma 15, $\bigcup_k E_k = E$.

We have the estimate (14) for each subsquare $Q' \in \mathcal{Q}_k$. The additivity of μ_Q across disjoint subsquares (10) implies that the probability of the event E_k under $\mu_Q^\#$ is a weighted average of the probabilities under $\mu_{Q'}^\#$, where Q' runs through the subsquares:

$$\mu_Q^\#(E_k) = \sum_{Q' \in \mathcal{Q}_k} \frac{\|\mu_{Q'}\|}{\|\mu_Q\|} \mu_{Q'}^\#(E_k)$$

By (14), we have

$$\mu_{Q'}^\#(E_k) \leq c r^{-\frac{1}{2}(4a-1)}$$

for each $Q' \in \mathcal{Q}_k$, with a constant c that depends only on κ . Taking a weighted average, we also have

$$\mu_Q^\#(E_k) \leq c r^{-\frac{1}{2}(4a-1)}.$$

Since $\mu_Q^\#(E_k) \uparrow \mu_Q^\#(E)$, the escape estimate for the entire curve γ under Q -length-biased chordal SLE_κ follows. \square

We now know that chordal SLE biased by the time spent in a small square Q is unlikely to escape far from a circle C around Q between the first and last times that it visits C .

We remark that the statements of Proposition 12 and Theorem 14 remain valid if Q is a rectangle rather than a square, with the same proofs.

The next section will show that the time spent while near the square is also small, with high probability.

5 Time spent near a small square

The main tools for bounding the time spent near a small square are the up-to-constant bounds on the two-point Green's function established by Lawler and Rezaei in [8]. Because the expected natural time spent by SLE in a region is simply the integral of the Green's function over that region, these bounds provide a simple first-moment bound on the time spent that suffices for the proof of Theorem 1.

Theorem 16. *If $\kappa < 8$, there exists $c < \infty$ such that the following holds. Let $D = (D, z, w)$ be a configuration and $U \subset D$ a disk such that $\text{dist}(U, \partial D) \geq 3 \text{diam}(U)$. Let γ be either two-sided radial SLE_κ through a point $\zeta \in U$ or chordal SLE_κ weighted by the time spent in a subset S of U . Then $\mathbf{E}[\Theta(U)] \asymp \text{diam}(U)^d$. In particular, $\mathbf{P}\{\Theta(U) > \epsilon\} < \epsilon$ as long as $\text{diam}(U) < c_1 \epsilon^{2/d}$, where c_1 is a constant depending only on κ .*

The theorem follows by a simple first moment estimate, which rests on the following lemma. Denote by $G(\zeta, \omega)$ the (Euclidean) unordered two-point Green's function for chordal SLE_κ as considered in [9],

$$G(\zeta, \omega) = \lim_{\epsilon, \delta \rightarrow 0} (\epsilon \delta)^{d-2} \mathbf{P}\{\text{dist}(\gamma, \zeta) < \epsilon, \text{dist}(\gamma, \omega) < \delta\}.$$

Lemma 17. *Under the hypotheses of Theorem 16,*

$$\mathbf{E}[\Theta(U)] = \frac{\int_U \int_U G(\zeta, \omega) dA(\omega) d\lambda(\zeta)}{\int_U G(\zeta) d\lambda(\zeta)},$$

where λ is a point mass at ζ in the case of two-sided radial SLE through ζ , or the area measure on the subset S in the weighted chordal case.

Proof. From Zhan's proof in [21] that chordal SLE_κ is reversible for $\kappa \leq 4$, it follows that two-sided radial SLE_κ is also reversible; that is, the reversal of two-sided radial SLE_κ in D from z to w through ζ has the law of two-sided radial SLE_κ in D from w to z through ζ . To use this fact, we decompose $\Theta(U)$ (for fixed U) as the sum of Θ_1 and Θ_2 , these being the natural time spent by the two-sided radial SLE_κ curve γ in U before and after visiting ζ , respectively. The law of $\gamma[\tau_\zeta, t_\gamma]$, conditional on the initial segment of γ up to the time τ_ζ that it hits ζ , is simply chordal SLE_κ in the slit domain from ζ to w . Hence the conditional expected time spent in U is

$$\mathbf{E}[\Theta_2 \mid \mathcal{F}_{\tau_\zeta}] = \int_U G_{D_{\tau_\zeta, \zeta, w}}(\omega) dA(\omega).$$

Taking an unconditional expectation,

$$\mathbf{E}[\Theta_2] = \int_U \mathbf{E}[G_{D_{\tau_\zeta, \zeta, w}}(\omega)] dA(\omega) = \int_U \frac{\hat{G}(\zeta, \omega)}{G(\zeta)} dA(\omega)$$

where $\hat{G}(\zeta, \omega)$ is the (Euclidean) ordered two-point Green's function for chordal SLE_κ [14, 9] in the configuration (D, z, w) . Applying reversibility

of two-sided radial SLE, we may similarly write

$$\mathbf{E}[\Theta_1] = \int_U \frac{\hat{G}(\omega, \zeta)}{G(\zeta)} dA(\omega).$$

Since the unordered Green's function is the sum of the two ordered Green's functions for that pair of points [14], the lemma follows in the two-sided radial case.

In the case of chordal SLE_κ weighted by the time spent in a subset S of U , which we denote μ_S , we have

$$\mathbf{E}_{\mu_S}[\Theta(U)] = \frac{\mathbf{E}[\Theta(U)\Theta(S)]}{\int_S G(\zeta) dA(\zeta)} = \frac{\int_S \int_U G(\zeta, \omega) dA(\omega) dA(\zeta)}{\int_S G(\zeta) dA(\zeta)}.$$

The last equality follows from the relation

$$\mathbf{E}[\Theta(A)^2] = \int_A \int_A G(\zeta, \omega) dA(\omega) dA(\zeta),$$

proved in [9], and linearity. \square

Proof of Theorem 16. We first recall the estimate on the unordered two-point Green's function that appeared in [8]. Namely, for chordal SLE_κ in \mathbb{H} from 0 to ∞ , if $\beta = 4a - 1 + d - 2 = \frac{\kappa}{8} + \frac{8}{\kappa} - 2$ and $|\zeta| < |\omega|$ with $q = |\zeta - \omega|/|\omega|$, then

$$G(\zeta, \omega) \asymp G(\zeta) G(\omega) q^{d-2} (S(\zeta) \vee q)^{-\beta}.$$

Substituting the known form of $G(\omega)$, we see that

$$G(\zeta, \omega) \asymp G(\zeta) |\zeta - \omega|^{d-2} \left(\frac{q}{S(\omega)} \vee 1 \right)^{-\beta}.$$

From the assumption that $\text{dist}(U, \partial D) \geq 3 \text{diam}(U)$, we can use distortion estimates to see that the last factor is of constant order in our case.

Return to considering SLE_κ in D from p to q . It is now simple to see that, for any $z \in U$,

$$\int_U G(\zeta, \omega) dA(\omega) \asymp G(\zeta) \int_U |\zeta - \omega|^{d-2} dA(\omega) \asymp \text{diam}(U)^d.$$

By Lemma 17 and the fact that $G(\zeta) \asymp G(\omega)$ for $\zeta, \omega \in U$, we see that under two-sided radial or weighted chordal SLE_κ as per the statement,

$$\mathbf{E}[\Theta(U)] \asymp \text{diam}(U)^d. \quad \square$$

6 Estimates on the distance between length-biased chordal SLE_κ and the aggregate of two-sided radial SLE_κ

The integral of measures is defined as the limit of its Riemann sum approximations as the mesh of the partition tends to zero (see Section 3). We shall prove directly that length-biased chordal SLE_κ is the limit of these Riemann sums.

To make this argument, we introduce a metric d_S on finite measures μ, ν on \mathcal{K} (the space of parametrized curves introduced in Section 3), given by

$$d_S(\mu, \nu) = \left| \log \frac{\|\mu\|}{\|\nu\|} \right| + \tilde{d}(\mu^\#, \nu^\#).$$

The advantage of this metric over the Prokhorov metric \tilde{d} introduced earlier is that it captures the notion of relative error and is stable (up to a constant) under addition. Indeed, we have the following lemma.

Lemma 18. *Suppose I is a finite set and, for $i \in I$, μ_i and ν_i are measures in \mathcal{K} . If $d_S(\mu_i, \nu_i) \leq \epsilon \leq 1$ for all $i \in I$, then*

$$d_S\left(\sum_i \mu_i, \sum_i \nu_i\right) \leq 18 \epsilon.$$

Proof. Write $\mu = \sum_i \mu_i$ and $\nu = \sum_i \nu_i$. It is clear that

$$\left| \log \frac{\|\mu\|}{\|\nu\|} \right| \leq \epsilon.$$

For any $i \in I$ and Borel set $A \subset \mathcal{K}$, we have $\mu_i^\#(A) \leq \nu_i^\#(A^\epsilon) + \epsilon$ and therefore

$$\mu_i(A) \leq \frac{\|\mu_i\|}{\|\nu_i\|} \nu_i(A^\epsilon) + \|\mu_i\| \epsilon.$$

Using that $\|\mu_i\| \leq e^\epsilon \|\nu_i\|$ and summing over i , we have the bound

$$\mu(A) \leq e^\epsilon \nu(A^\epsilon) + \|\mu\| \epsilon \leq \nu(A^\epsilon) + c \|\mu\| \epsilon$$

which translates readily into the following bound for the probability measure:

$$\mu^\#(A) \leq \nu^\#(A^\epsilon) e^\epsilon \frac{\|\mu\|}{\|\nu\|} + \epsilon \leq \nu^\#(A^\epsilon) e^{2\epsilon} + \epsilon \leq \nu^\#(A^\epsilon) + 17 \epsilon,$$

where the last inequality uses that $e^{2\epsilon} - 1 \leq 2\epsilon(e^2 - 1) < 16\epsilon$. Swapping μ and ν , we conclude that $\tilde{d}(\mu^\#, \nu^\#) \leq 17\epsilon$. \square

The key idea to the proof is that, as suggested by Proposition 12, there is not much difference between chordal SLE_κ weighted by the time spent in a square Q and two-sided radial SLE through a point $\zeta \in Q$, at least for the initial and final segments away from that square. The remaining segment connecting the initial and terminal segments is short (in space and time) with high probability. We assemble these results in the following proposition.

Proposition 19. *If $\kappa \leq 4$, there exist $\alpha > 0$, $c < \infty$ such that the following holds. Let Q be a square of side length ϵ at distance at least $\delta \in (0, 1]$ from the boundary of a domain D , and let μ be chordal SLE_κ in D . Let ζ, ζ' be points in Q . Assume that $\epsilon < \delta/100$. Then*

$$d_S(\mu_Q, \mu_\zeta A(Q)) \leq c(\epsilon/\delta)^\alpha.$$

In particular, $d_S(\mu_{\zeta'}, \mu_\zeta) \leq c(\epsilon/\delta)^\alpha$.

Proof. By scaling the given configuration up, it suffices to assume that $\delta = 1$. (Observe that distance and time each increase by a fixed factor under scaling up and hence the \tilde{d} distance between curves can only increase).

By (13) we have

$$\left| \log \frac{\|\mu_Q\|}{\|\mu_\zeta\| A(Q)} \right| \leq c\epsilon.$$

Now let C be the circle around Q of radius $\epsilon^{1/2}$ and recall the corresponding stopping times τ_C for γ and $\tilde{\tau}_C$ for $\tilde{\gamma}$. Considering the probability measures as measures solely on the pair $(\gamma_\tau, \tilde{\gamma}_{\tilde{\tau}})$, by (12) we have

$$\left| \log \frac{d\mu_Q^\#}{d\mu_\zeta^\#} \right| \leq c\epsilon^{1/2}.$$

We must address the behaviour of the intervening segment $\gamma[\tau_C, t_\gamma - \tilde{\tau}_C]$: conditional on $(\gamma_\tau, \tilde{\gamma}_{\tilde{\tau}})$, the intervening segment of γ is unlikely to go far in space or time.

By Theorem 14, the probability under $\mu_Q^\#$ that $\gamma[\tau_C, t_\gamma - \tilde{\tau}_C]$ leaves the circle C' of radius $\epsilon^{1/4}$ around Q is at most $c\epsilon^{(4a-1)/8}$. By [4, Theorem 1.3], the same estimate holds under two-sided radial SLE_κ , $\mu_\zeta^\#$.

Finally, by Theorem 16, the probability (under either $\mu_Q^\#$ or $\mu_\zeta^\#$) that more than $c\epsilon^{d/8}$ time is spent inside C' is at most $c\epsilon^{d/8}$, with a constant c depending only on κ . (Note that the square Q here is actually the set S in the application of Theorem 16.)

Choose $\alpha = \min(4a - 1, d)/8 < 1/4$. Denote $\gamma_2 := \gamma[\tau_C, t_\gamma - \tilde{\tau}_C]$ and let $\hat{\gamma}_2$ be the segment $[\gamma(\tau_C), \tilde{\gamma}(\tilde{\tau}_C)]$ (traversed over the same time interval of length $t_\gamma - \tilde{\tau}_C - \tau_C$). We have, therefore, under both $\mathbf{P} = \mu_Q^\#$ and $\mathbf{P} = \mu_\zeta^\#$,

$$\mathbf{P}\{\tilde{d}(\gamma_2, \hat{\gamma}_2) > c\epsilon^\alpha \mid \gamma_{\tau_C}, \tilde{\gamma}_{\tilde{\tau}_C}\} < c\epsilon^\alpha.$$

Since γ is the concatenation of $\gamma_1 := \gamma_{\tau_C}$, γ_2 and the reversal γ_3 of $\tilde{\gamma}_{\tilde{\tau}_C}$, it follows from (8) that

$$\mathbf{P}\{\tilde{d}(\gamma, \gamma_1 \oplus \hat{\gamma}_2 \oplus \gamma_3) > c\epsilon^\alpha \mid \gamma_{\tau_C}, \tilde{\gamma}_{\tilde{\tau}_C}\} < c\epsilon^\alpha.$$

Finally, the Radon-Nikodym derivative estimates above show that we may couple the measures $\mu_Q^\#$ and $\mu_\zeta^\#$ on initial and final segments (γ_1, γ_3) and (η_1, η_3) under a joint law \mathbf{P} such that

$$\mathbf{P}\{\gamma_1 \oplus \hat{\gamma}_2 \oplus \gamma_3 \neq \eta_1 \oplus \hat{\eta}_2 \oplus \eta_3\} < c\epsilon^\alpha.$$

In this coupling \mathbf{P} , conditional on $\gamma_1, \gamma_3, \eta_1, \eta_3$, let us declare the intermediate segments γ_2 and η_2 to be independent and follow the laws of Q -length-biased chordal SLE_κ and two-sided radial SLE_κ through ζ in the doubly slit domains $D \setminus \{\gamma_1, \gamma_3\}$ and $D \setminus \{\eta_1, \eta_3\}$ respectively. Then the completed curves $\gamma = \gamma_1 \oplus \gamma_2 \oplus \gamma_3$ and $\eta = \eta_1 \oplus \eta_2 \oplus \eta_3$ are marginally distributed as Q -length-biased chordal SLE_κ and two-sided radial SLE_κ through ζ in D , respectively. Adding the above estimates yields the estimate

$$\begin{aligned} \mathbf{P}\{\tilde{d}(\gamma, \eta) > 2c\epsilon^\alpha\} &\leq \mathbf{P}\{\tilde{d}(\gamma, \gamma_1 \oplus \hat{\gamma}_2 \oplus \gamma_3) > c\epsilon^\alpha\} \\ &\quad + \mathbf{P}\{\tilde{d}(\eta, \eta_1 \oplus \hat{\eta}_2 \oplus \eta_3) > c\epsilon^\alpha\} \\ &\quad + \mathbf{P}\{\gamma_1 \oplus \hat{\gamma}_2 \oplus \gamma_3 \neq \eta_1 \oplus \hat{\eta}_2 \oplus \eta_3\} \leq 3c\epsilon^\alpha, \end{aligned}$$

and hence $\tilde{d}(\mu_Q^\#, \mu_\zeta^\#) < 3c\epsilon^\alpha$, as required. \square

Proof of Theorem 1. We first establish that the proper Riemann integrals that appear in Definition 5 over the truncated domain $D_{r,s,t} = D \cap B(0, \frac{1}{r}) \setminus \overline{B(z, s)} \setminus \overline{B(w, t)}$ are $D_{r,s,t}$ -length-biased chordal SLE_κ :

$$\int_{D_{r,s,t}} f(\zeta) dA(\zeta) = \mu_{D_{r,s,t}}. \quad (15)$$

The use of $B(0, \frac{1}{r})$ is unnecessary for the theorem as stated, but see Remark 20.

Let $\epsilon \in (0, 1)$ be arbitrary. Consider a rectangle R containing $D_{r,s,t}$ and a tagged partition \mathcal{P} of R with mesh $\|\mathcal{P}\| < \epsilon$ and arbitrary sample points $z_Q \in Q$ for each $Q \in \mathcal{P}$.

We shall divide the subrectangles of \mathcal{P} into two groups. Let I (the “interior” rectangles) consist of those subrectangles $Q \in \mathcal{P}$ that lie at distance $\epsilon^{1/2}$ or greater from $\mathbb{C} \setminus D_{r,s,t}$, and let $J = \mathcal{P} \setminus I$.

For each $Q \in I$, we have $d_S(\mu_Q, \mu_{z_Q} A(Q)) \leq c \epsilon^{\alpha/2}$ by Proposition 19. It follows by Lemma 18 that

$$d_S\left(\mu_I, \sum_{Q \in I} \mu_{z_Q} A(Q)\right) \leq c \epsilon^{\alpha/2},$$

where $\mu_I = \sum_{Q \in I} \mu_Q$ by (10). (By a convenient abuse of notation, we write μ_I for $\mu_{\bigcup_{Q \in I} Q}$.)

We also know the total mass of μ_I is $\int_I G_D(\zeta) dA(\zeta)$, which is of order 1 for ϵ small enough. (Recall that the configuration D is fixed throughout this proof.) This is because G_D is integrable on $D_{r,s,t}$.

We may therefore convert the d_S estimate into the following estimate on the Prokhorov distance between the measures in question:

$$\tilde{d}\left(\mu_I, \sum_{Q \in I} \mu_{z_Q} A(Q)\right) \leq c_D \epsilon^{\alpha/2},$$

where c_D is a constant that depends only on D (and κ , since our generic constants c may depend on κ).

Let us now consider the remaining subrectangles $Q \in J$. The key observation here is that, even if we do not control the curve’s behaviour under μ_Q and μ_{z_Q} , the total mass of these measures is insufficient to make a significant contribution to the total. Indeed, since $D_{r,s,t}$ has bounded, piecewise analytic boundary, the total area of the rectangles in J that actually intersect $\overline{D}_{r,s,t}$ is at most $c_{D_{r,s,t}} \epsilon^{1/2}$, where $c_{D_{r,s,t}}$ depends only on $D_{r,s,t}$. The total masses

$$\|\mu_J\| = \int_{D_{r,s,t} \cap \bigcup_{Q \in J} Q} G_D(\zeta) dA(\zeta)$$

and

$$\left\| \sum_{Q \in J} \mu_{z_Q} 1_{z_Q \in D_{r,s,t}} A(Q) \right\| = \sum_{Q \in J: z_Q \in D_{r,s,t}} G_D(z_Q) A(Q)$$

are both at most $c_{D_{r,s,t}} \epsilon^{1/2}$.

Using the additivity of the Prokhorov metric \tilde{d} , as seen in (9), we may conclude that

$$\tilde{d}\left(\mu_I + \mu_J, \sum_{Q \in \mathcal{P}} \mu_{z_Q} 1_{z_Q \in D_{r,s,t}} A(Q)\right) \leq c_{D_{r,s,t}} \epsilon^{\alpha/2},$$

and since $\mu_{D_{r,s,t}} = \mu_I + \mu_J$, we have proved (15).

Now we need to take an increasing limit of the improper integrals in (15). We will need to use one more fact about the Green's function, namely that

$$\int_D G_D(\zeta) dA(\zeta) < \infty. \quad (16)$$

Observe that G_D is continuous on \overline{D} except at z and w , and is locally integrable at z and w . This follows, after applying a conformal map, from the integrability of $G_{\mathbb{H}}$ near 0 and the fact that $G_{\mathbb{H}}$ is invariant under the inversion $z \mapsto -1/z$. Second, G_D is continuous on \overline{D} except at z and w , with value 0 on $\partial D \setminus \{z, w\}$. As D is bounded, we have (16).

We may now simply note that

$$\mu_D = \mu_{D_{r,s,t}} + \mu_{D \setminus D_{r,s,t}} = \int_{D_{r,s,t}} f(\zeta) dA(\zeta) + \mu_{D \setminus D_{r,s,t}}$$

and therefore, by (16),

$$\tilde{d}\left(\mu_D, \int_{D_{r,s,t}} f(\zeta) dA(\zeta)\right) \leq \|\mu_{D \setminus D_{r,s,t}}\| = \int_{D \setminus D_{r,s,t}} G_D(\zeta) dA(\zeta) \rightarrow 0$$

as $r, s, t \rightarrow 0$. Therefore length-biased chordal SLE_κ μ_D is equal to the improper integral

$$\mu_D = \int_D G_D(\zeta) dA(\zeta). \quad \square$$

We conclude with some remarks on the proof.

Remark 20. The case where D is unbounded can be handled by the same improper integral mechanism, but we do need to verify (16). Suppose still that D has analytic boundary, including at ∞ , meaning that $\partial(1/D)$ is analytic at 0. The question is whether G_D is integrable at ∞ . This is certainly false if z or w is ∞ . Otherwise, the following computation (in the case

$(D, z, w) = (\mathbb{H}, x, y)$ for simplicity, though the result is the same for any D can be used to verify that this will be the case if and only if $4a > \sqrt{2} + 1$, that is, if and only if $\kappa < 8(\sqrt{2} - 1) = 3.3137\dots$:

$$\begin{aligned} \int_{\zeta \in \mathbb{H}: |\zeta| > R^{-1}} G_{\mathbb{H}, x, y}(\zeta) dA(\zeta) &= \int_{z \in \mathbb{H}: |z| < R} G_{\mathbb{H}, -1/x, -1/y}(z) |z|^{2(2-d)} \frac{dA(z)}{|z|^4} \\ &\asymp c_{x, y, \kappa} \int_{z \in \mathbb{H}: |z| < R} (\operatorname{Im} z)^{4a+d-3} |z|^{-2d} dA(z) \\ &\asymp c_{x, y, \kappa} \int_0^R r^{4a - \frac{1}{4a} - 3} dr. \end{aligned}$$

Remark 21. If we allow D to have a non-analytic boundary, the result is still true so long as we can prove the “discarded” parts of the measure are still small. In particular, if G_D is integrable and the area of the $\epsilon^{1/2}$ -expansion $(\partial D)^{\epsilon^{1/2}}$ decays as a power of ϵ as $\epsilon \rightarrow 0$, we can still apply the same proof with a different power of ϵ . This certainly encompasses such cases as bounded domains with piecewise C^1 boundaries.

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